Constraint Based Computation of Periodic Orbits of Chaotic Dynamical Systems

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Dynamical system: State evolves with time with deterministic rule
Before: Complex dynamics come from complex dynamical systems
Chaos theory: Very simple systems can be chaotic!
Chaos Is Everywhere

- Dynamical system: State evolves with time with deterministic rule
- Before: Complex dynamics come from complex dynamical systems
- Chaos theory: Very simple systems can be chaotic!

\[ x_{k+1} = f(x_k) \]: Discrete time, continuous state

\[(0, 0, 1, 0, \ldots) \xrightarrow{\sigma} (0, 1, 0, \ldots)\]

Discrete time, discrete state (symbolic dynamics)
History (subjective and non exhaustive)

- Late 19th century (in France :-)
  - Jacques Hadamar (symbolic dynamics on hyperbolic billards)
  - Henri Poincaré (chaos in the three-body problem, Poincaré maps)
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  - James A. Yorke (period three theorem)
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- **Modern analysis**

![]()

ODE \[\rightarrow\] Poincaré map  
Discrete time & continuous state \[\rightarrow\] Markov partition  
Symbolic dynamics on discrete time & discrete state
Chaos in Continuous and Discrete Time Dynamical Systems

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\[ \text{ODE} \quad \longrightarrow \quad \text{Discrete time & continuous state} \quad \longrightarrow \quad \text{Markov partition} \]

\[ \text{Symbolic dynamics on discrete time & discrete state} \]

Discrete Time Dynamical Systems

- \( X \subseteq \mathbb{R}^n \) and \( F : X \to X \)
- Forward orbit of \( x \in X \):
  \((x, f(x), f(f(x)), \ldots) = (f^k(x))_{k \in \mathbb{N}}\)

\((\text{Hénon map})\)
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![Diagram](image)

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(Hénon map)
Definition of Chaos

- Exponential sensitivity to initial conditions: Wrong common sense
  ⇒ E.g. \( f : \mathbb{R} \to \mathbb{R}, f(x) = 2x \) (exponentially sensitive, but non chaotic)
Hyperbolic Chaos and Periodic Orbits

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  ⇒ E.g. $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = 2x$ (exponentially sensitive, but non chaotic)
- Closer to true: $X$ is bounded, and $f$ exponentially sensitive to initial conditions
  ⇒ E.g. $f : [0, 1] \rightarrow [0, 1]$, $f(x) = 2x \mod 1$ (the modulo 2 map)
- Both expanding and contracting $\Rightarrow$ hyperbolic (roughly speaking)
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- Most common definition of chaos:
  \[ f \text{ chaotic} \iff h_{\text{top}}(f) > 0 \text{ (i.e. strictly positive topological entropy\(^a\)} \]

\(^a\)The topological entropy is a number that quantifies the exponential divergence
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Periodic Orbits

- \( (f^k(x))_{k \in \mathbb{Z}} \) is \( n \)-periodic \( \iff f^{k+n}(x) = f^k(x) \iff x = f^n(x) \)
- \( P_n(f) = \) number of \( n \)-periodic orbits
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- Key role in hyperbolic chaotic systems: \( P_n(f) \approx O(e^{n h_{\text{top}}(f)}) \)
# Hyperbolic Chaos and Periodic Orbits

## Definition of Chaos

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## Periodic Orbits

- $(f^k(x))_{k \in \mathbb{Z}}$ is $n$-periodic $\iff f^{k+n}(x) = f^k(x) \iff x = f^n(x)$
- $P_n(f) = \text{number of } n\text{-periodic orbits}$
- Key role in hyperbolic chaotic systems: $P_n(f) \approx O(e^{nh_{\text{top}}(f)})$

### Counting periodic orbits

\[
\text{Counting periodic orbits} \implies \text{estimation of } h_{\text{top}}(f) \approx \frac{\log P_n(f)}{n}
\]
The Contribution

- Z. Galias (2001): Dedicated algorithm based on bisection and interval Newton for rigorously computing (hence counting) periodic orbits
- Numerical constraint programming:
  - Easier modeling
  - Tunable search and resolution strategies
  - Asymptotic performance gain w.r.t. Galias (although not critical, the resolution process is intrinsically exponential)
  - Available solvers
Outline

1. Topological Entropy: An Introducing Example
2. Periodic Orbits as CSPs
3. Experiments
4. Conclusion
The Modulo 2 Map (also called modulo 2 map)

Map Definition

- $X = [0, 1]$ and $f(x) = 2x \mod 1$
- Most simple chaotic dynamical system
- Orbit of $x_0 = 0.1$
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  - $x_1 = 0.2$

$\Rightarrow$ Converged to a periodic orbit

Symbolic Representation

- $0.b_1b_2b_3\cdots$ → binary representation of $x \in X$
- $f(0.b_1b_2b_3\cdots) = 0.b_2b_3\cdots \Rightarrow f$ is the shift map on sequences of two symbols
  - $(10) = 0.0001100110011\cdots$
  - $(2) = 0.001100110011\cdots$ (periodic)

- More generally:
  - $x \in \mathbb{Q} \iff$ binary representation periodic after some fixed index
  - $x \in \mathbb{Q} \Rightarrow (f^k(x))_k \in \mathbb{N}$ converge to a periodic orbit after a finite number of steps
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The figure illustrates the modulo 2 map with a graph showing the orbit of $x_0 = 0.1$.
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  - \( x_3 = 0.8 \)

Goldsztejn, Granvilliers and Jermann

Periodic Orbits of Chaotic Dynamical Systems

Uppsala (Sweden), Sept. 16-20, 2013
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The Modulo 2 Map: Sensitivity to Initial Conditions

Exponential Sensitivity

- Long term simulation impossible
  - Finite precision computation (e.g. IEEE double): $x = b_1 b_2 \cdots b_{64} 000 \cdots$
  - Converge to 0 after finite number of iterations
- Exponential divergence
  - $x = b_1 b_2 \cdots$ and $y = b_1 b_2 \cdots b_{K-1} |1 - b_K| c_{K+1} \cdots \Rightarrow d(x, y) \leq \frac{1}{2^K}$
  - $f^K(x) = b_K \cdots$ and $f^K(y) = |1 - b_K| \cdots \Rightarrow d(f^K(x), f^K(y)) \geq \frac{1}{2}$
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  - \( f^K(x) = b_K \cdots \) and \( f^K(y) = |1 - b_K| \cdots \) \( \Rightarrow \) \( d(f^K(x), f^K(y)) \geq \frac{1}{2} \)

### Topological Entropy

- \( s(n, \epsilon) \equiv \text{maximal cardinality of a set whose elements can be separated of } \epsilon \text{ by at most } n \text{ iterations of the map} \)
- \( s(n, \epsilon) \) grows exponentially with \( n \) \( \Rightarrow \) exponential divergence
- \( \Rightarrow \) \( s(\epsilon) := \limsup_{n \to \infty} \frac{\log s(n, \epsilon)}{n} > 0 \) \( \Rightarrow \) exponential divergence

\[
h_{\text{top}}(f) = \limsup_{\epsilon \to 0} s(\epsilon)
\]

- **Exponential divergence** \( \iff \) \( h_{\text{top}}(f) > 0 \) \( \text{def.} \) chaotical
Topological Entropy of the Modulo 2 Map

Lower Bound Through Periodic Orbits

- $E_n \subseteq [0, 1]$: Points whose orbit is $n$-periodic
- $\text{card} E_n = 2^n$
Lower Bound Through Periodic Orbits

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- $x \neq y \in E_n$ have one bit different $\Rightarrow$ separated of 0.5 by at most $n$ iteration of $f$
  $\Rightarrow s(n, 0.5) \geq 2^n$
  $\Rightarrow h_{\text{top}}(f) \geq \frac{\log 2^n}{n} = \log 2$ (in fact, $h_{\text{top}}(f) = \log 2$)
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Simple Map, Amazing Complexity

- $P_n = 2^n$ (think of $n = 200$)
- Infinitely many periodic orbits, which are interleaved in a complex structure (exponential divergence)
- Non-periodic orbits seem random (e.g. $\pi$)
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2. Periodic Orbits as CSPs
   - CSP Models for Periodic Orbits Computation
   - CSP Resolution: Search and Resolution Strategies
3. Experiments
4. Conclusion

Constraint Programming

Strengths:
- Separation of modeling and solving
- Many strategies for searching and filtering that can be tuned
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Folded Model

- $d$ variables: $x = (x_1, \ldots, x_d) \in X \subseteq \mathbb{R}^d$
- $d$ constraints: $f^n(x) = x$ (vectorial equality $\Rightarrow$ $d$ equality constraints)
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- Advantage: Number of variables do not depend on $n$
- Disadvantage: Constraints get more and more complex as $n$ increases
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$\Rightarrow$ Useless except for very simple systems (Galias, and confirmed by our experiments)
CSP Models for \( n \)-Periodic Orbits Computation (1/2)

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Unfolded (functional) Model

- \( d \times n \) variables: \( x_k = (x_{k1}, \ldots, x_{kd}) \in X \subseteq \mathbb{R}^d \), \( k \in \{0, \ldots, n-1\} \)
- \( d \times n \) constraints:
  - \( f(x_k) = x_{k+1}, k \in \{0, \ldots, n-2\} \) (links between states at different times)
  - \( f(x_{n-1}) = x_0 \) (periodic orbit)
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- Disadvantage: Number of variables increase linearly with $n$ (search space increase exponentially)
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$\Rightarrow$ Much better than folded model (Galias, and confirmed by our experiments)
Sometimes: $f(x) = y \iff g(x, y) = 0$, where $g$ is simpler.
Unfolded (relational) Model

- Sometimes: $f(x) = y \iff g(x, y) = 0$, where $g$ is simpler
- $d \times n$ variables: $(x_{k1}, \ldots, x_{kd}) \in X \subseteq \mathbb{R}^d$, $k \in \{0, \ldots, n-1\}$
- $d \times n$ constraints:
  - $g(x_k, x_{k+1}) = 0$, $k \in \{0, \ldots, n-2\}$ (links between states at different times)
  - $g(x_{n-1}, x_0) = 0$ (periodic orbit)
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Additional Constraints

- Periodic orbits have a cyclic symmetry \( \Rightarrow \) RLEX partial symmetry breaking\(^a\):
  \[ x_{0,0} \leq x_{0,k}, k \in \{0, \ldots, n-1\} \]

\[ \Rightarrow \] Critical importance for solving long period problems
- Non-wandering set a priori enclosure \( \Rightarrow \) reduce the search space
  - Difficult to tune preprocessing
  - Our experiments: Useless because of constraint propagation, which quickly infer the information

\(^a\) Although partial, complete in practice for this problem (verified a posteriori for correct counting)
Standard Algorithm

- Variable domains are continuous
- General algorithm:
  - Filtering (HC4-revise, BC3-revise, CID, Newton)
  - Branching (maxdom on all or some variables)
  - Post processing: Validation of existence/uniqueness and symmetry breaking

⇒ Compute and certify all solutions $\rightarrow P_n$
### CSP Resolution: Search and Resolution Strategies

#### Standard Algorithm

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\[ \Rightarrow \text{Compute and certify all solutions } \rightarrow P_n \]

#### Post Processing Partial Symmetry Breaking Constraints

- RLEX is complete provided \( \text{card argmin}_{i \in \{1, n-1\}} x_{0,i} = 1 \)
- Numerically, no difference between
  - RLEX not complete and broken
  - RLEX complete, but close to not complete, and not broken (in this case, counting solutions is not sound)
- Need to check a posteriori that \( \overline{x}_{0,0} < x_{0,k}, k \in \{1, n-1\} \)
Unidimensional Maps: The Modulo 2 and Logistic Maps

\[
\begin{align*}
\quad P_n & \approx O(e^{0.69n}) \\
\quad t_{\blacksquare}(n) & \approx O(e^{0.74n}) \\
\quad t_{\triangle}(n) & \approx O(e^{0.73n}) \\
\quad t_{\triangleup}(n) & \approx O(e^{0.76n}) \\
\quad t_{\bullet}(n) & \approx O(e^{0.72n})
\end{align*}
\]

BC5 with \textit{maxdom} for all experiments; \quad \circ \equiv P_n; \quad \blacksquare \equiv \text{factorized logistic and modulo 2 unfolded models}; \quad \triangle \equiv \text{modulo 2 folded model}; \quad \triangleup \equiv \text{factorized logistic folded model}; \quad \bullet \equiv \text{non-factorized logistic unfolded model}.

Comments

- Log scale: Straight line \equiv\text{exponential (slope=exponential growth)}
- \(P_n\) shows the best asymptotic one can expect
- Asymptotic behavior can be read on small orbits problems
- Different models and tuning do not show drastic difference
The Hénon Map

\[ t \approx O(e^{0.46n}) \]

Comments

- \( P_n \) shows the best asymptotic one can expect: \( t_o(n) \approx O(e^{0.46n}) \)
- Again, asymptotic behavior can be read on small orbits problems
- Different models and tuning now show drastic differences
- Unfolded model with CID(3)+maxdom: \( t_{\Box}(n) \approx O(e^{0.51n}) \)
- Galias: \( t_G(n) \approx O(e^{0.58n}) \) (timings not meaningful)
The Ikeda Map

Comments

- $P_n$ shows the best asymptotic one can expect: $t_\circ(n) \approx O(e^{0.60n})$
- Again, asymptotic behavior can be read on small orbits problems
- Different models and tuning now show drastic differences
- Unfolded relational model with CID(9)+maxdom: $t_{\Box}(n) \approx O(e^{0.66n})$
Conclusion

Advantages of the CP Framework

- Flexible modeling and resolution strategies, tunable on easy instances
- Available solvers
- Asymptotic gain wrt Galias (although not critical wrt intrinsic exponential complexity)
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Discussion wrt Galias

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Future Work

- Higher dimensional systems
- ODE using Poincaré maps
- Accurate topological entropy lower bound