Reduction of CSP Dichotomy to Digraphs

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Marcel Jackson LA TROBE UNIVERSITY

Jakub Bulín (Charles U), Dejan Delić (Ryerson U),
Todd Niven (La Trobe U and Monash U).

Outline

Fixed template CSPs

- Constraint language: $\Gamma = (D; R_1, \ldots, R_m)$
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- *Usually D is finite. We will require the relation list to be finite.*

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Example

A directed graph $\Gamma = (D; \to)$, with $\to$ binary

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$$\begin{pmatrix} d_{1,1} \\ \vdots \\ d_{k,1} \end{pmatrix}, \begin{pmatrix} d_{1,2} \\ \vdots \\ d_{k,2} \end{pmatrix}, \ldots, \begin{pmatrix} d_{1,n} \\ \vdots \\ d_{k,n} \end{pmatrix} \in R \; \Rightarrow \begin{pmatrix} f(d_{1,1}, \ldots, d_{1,n}) \\ \vdots \\ f(d_{k,1}, \ldots, d_{k,n}) \end{pmatrix} \in R$$

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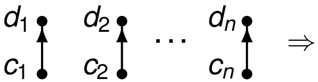

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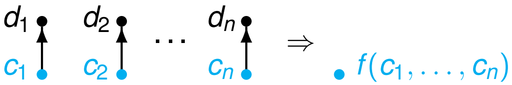

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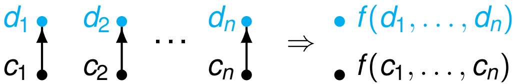

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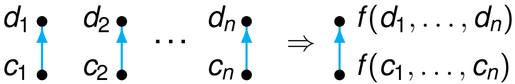

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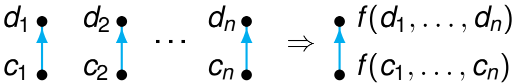

Example

- “automorphism” = “bijective unary polymorphism”
- “endomorphism” = “unary polymorphism”

Polymorphisms and complexity

- Polymorphisms determine computational complexity
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Problem	Polymorphism Algebra
HORNSAT	$\begin{array}{c\|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$

The Dichotomy Conjecture

Feder & Vardi 1994

Conjecture: For all Γ, $CSP(\Gamma)$ is either in $\mathtt{P}$ or is $\mathtt{NP}$-complete

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Algebraic conjectures

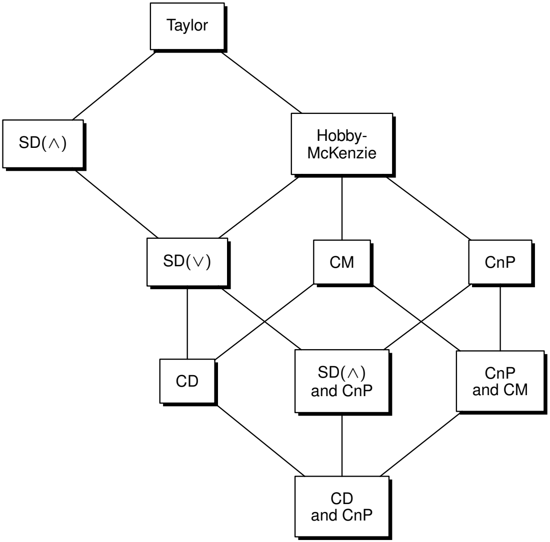
- Algebraic dichotomy conjecture: a characterisation of tractable CSPs
  - Bulatov, Jeavons, Krokhin
- conjectured/known polymorphism classifications of $\mathtt{NL}$, $\mathtt{L}$, and amenability to specific kinds of algorithms

Outline

Main Result

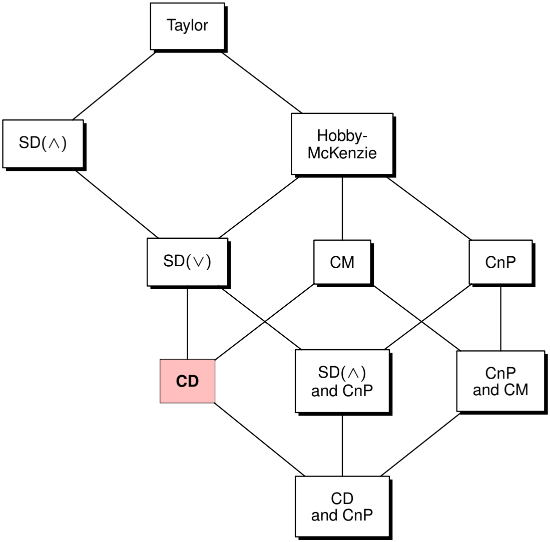
Every *CSP*(Γ) is *logspace* equivalent to a CSP over a balanced directed graph with (almost) the same polymorphism properties

- future investigations can be restricted to the special case of digraphs with no loss of generality



Polymorphism landscape

Each node depicts an algebraic property relevant to CSPs



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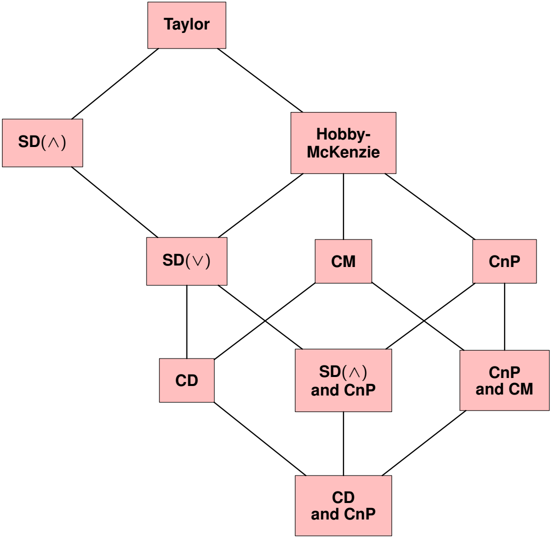
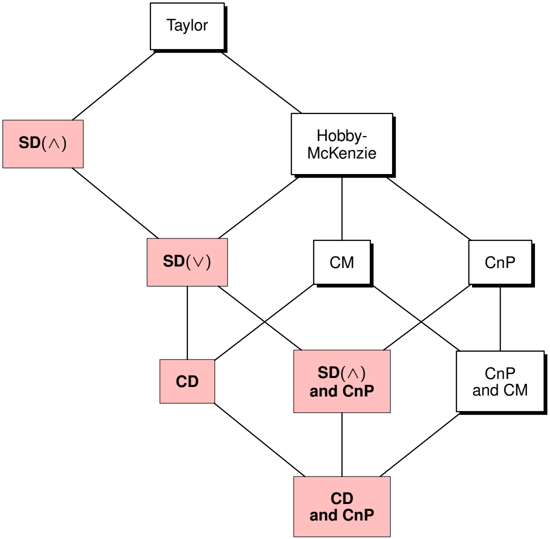
$$x = \nu(y, x, x, \dots, x) = \nu(x, y, x, \dots, x) = \dots = \nu(x, x, \dots, x, y)$$



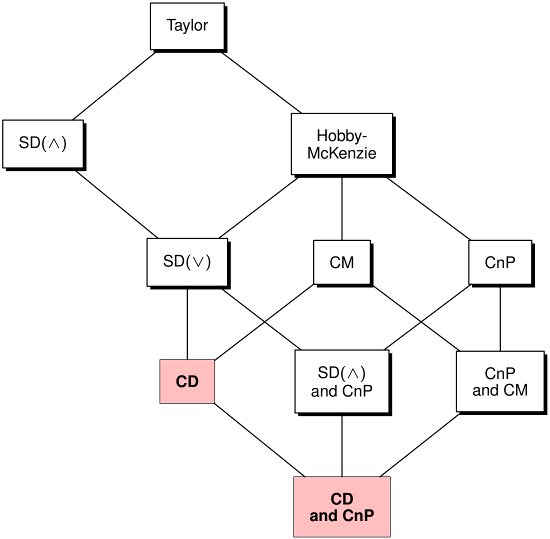
Polymorphism landscape

Algebraic dichotomy conjecture: $CSP(\Gamma)$ is tractable if it satisfies "Taylor" and is NP-complete otherwise



Polymorphism landscape

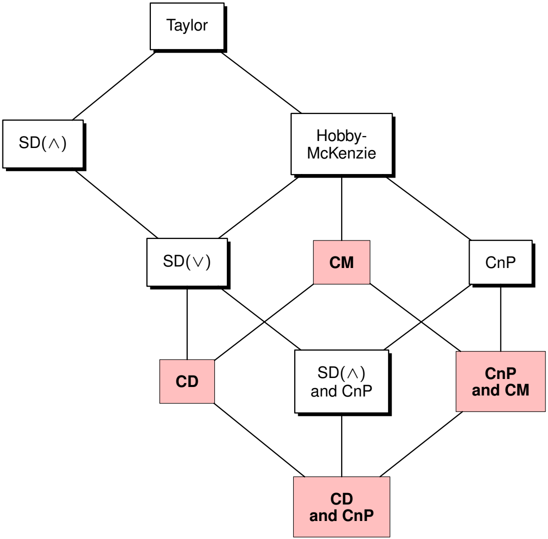
CSPs solvable by local consistency check (Barto and Kozik).



Polymorphism landscape

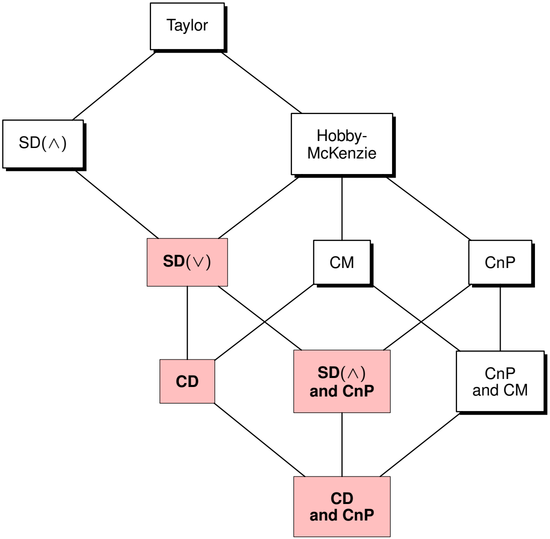
Strict width.

(Many authors)



Polymorphism landscape

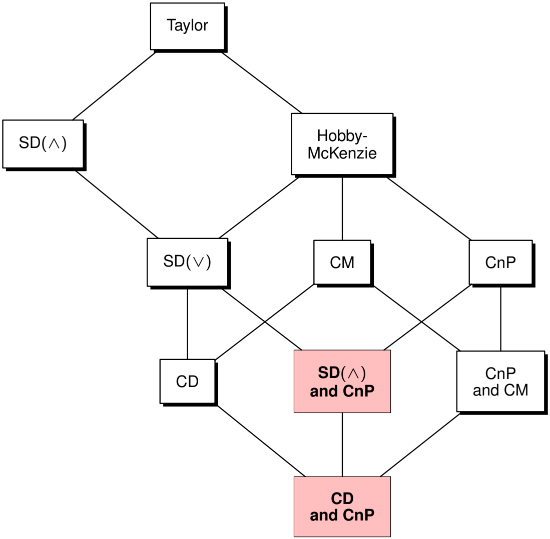
CSPs solvable by generalised Gaussian elimination algorithm (Barto + Berman, Idziak, Marković, McKenzie, Valeriote, Willard)



Polymorphism landscape

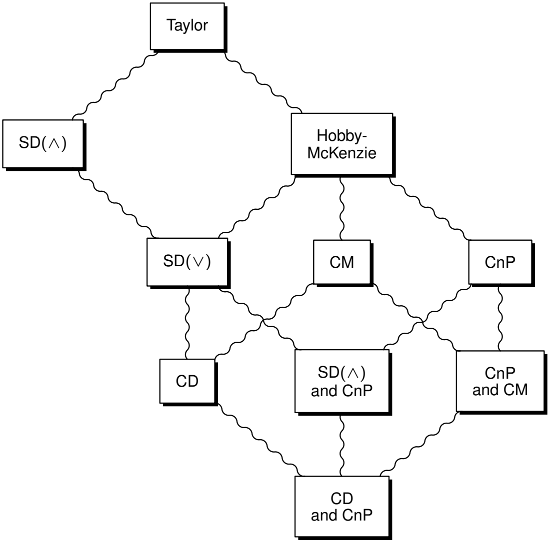
Algebraic $\mathtt{NL}$ conjecture:

Solvability in nondeterministic logspace??



Polymorphism landscape

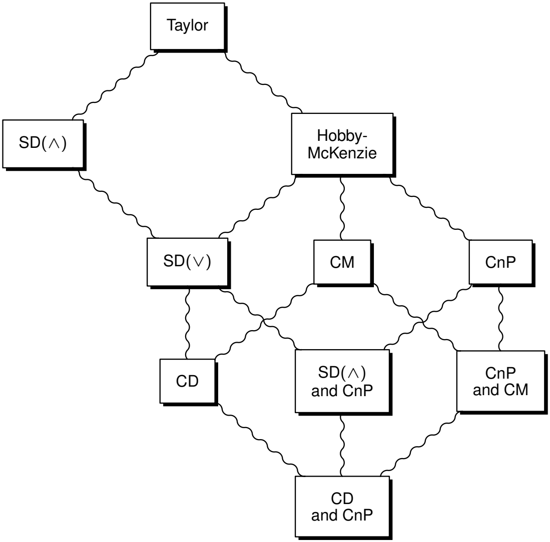
Algebraic L conjecture: Solvability in logspace??



Polymorphism landscape

Corollary of main result:

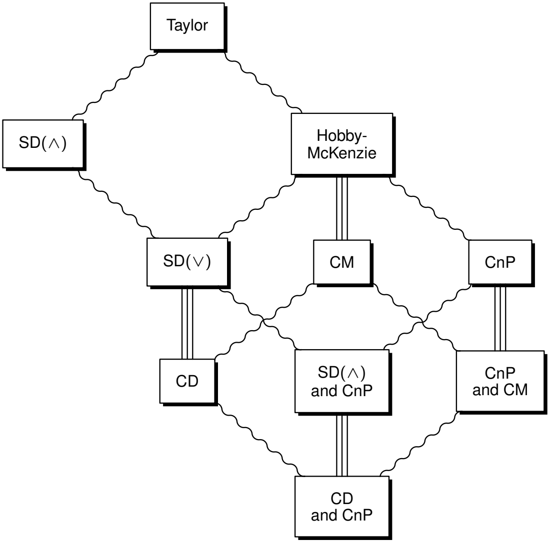
- *All regions can be distinguished using digraph CSPs*



Polymorphism landscape

Corollary of main result:

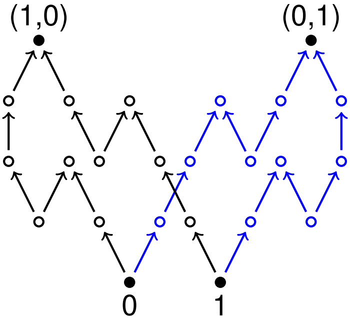
- *All regions can be distinguished using digraph CSPs*
- *The conjectures can be resolved by considering digraph CSPs only*



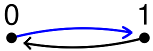
Polymorphism landscape

Note:
It is possible to find other logspace translations that obliterate some of these distinctions

Outline



(*Above.*) *The digraph obtained by applying the construction to the following two element cyclic digraph*



What *can't* be done

- The landscape of list homomorphism problems over digraphs is substantially restricted (Hell, Rafiey).
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Extensions

- The main result *does* neverthe-more-or-less apply to infinite domain CSPs (with finitely many relations).
- Approximation CSPs? Counting CSPs? Surjective CSPs? . . . CSPs?