Tractable Combinations of Global Constraints

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Global vs. table constraints

CSP solvers allow us to represent constraints *implicitly*, using built-in algorithms.

Constraint that specifies how often each domain value is to occur in a set of variables:

V	X	Y	Z
a	b	b	a
a	a	b	b
a	b	a	b
	...		

As GCC:

$K(a) = 2$
$K(b) = 2$

Global constraints

Many different global constraints exist:

- Global Cardinality (different kinds)
- AllDifferent
- NValue
- Regular language membership
- *SAT clauses*!

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A lot is known about propagation for such constraints.

Far less is known about *problem tractability*.

Structural tractability of global constraints

For the extensional case, we can get tractable classes by bounding

- treewidth
- query width
- hypertree width
- fractional hypertree width

For problems with global constraints, only treewidth works.

Proof: Take k-colourability as CSP, add a cardinality constraint on V with $K(a) = \{0, \ldots, |V|\}$.

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For problems with global constraints, only treewidth works.

Bounded treewidth implies bounded arity, so not very useful.

Is there a class between treewidth and query width?

Interlude: What is a global constraint?

A *global constraint* is a parameterized poly-time algorithm.

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It takes a description, and decides which assignments are allowed.

This covers the known global constraints.

Various similar definitions exist in the literature.

Discovering a new class

Global constraints can have exponentially many solutions in their size.

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A SAT clause disallows exactly *one assignment*.

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For a set of overlapping SAT clauses, I only need to care about each forbidden assignment. The rest are equivalent.

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If the clauses overlap on some set X of variables, I need only consider projections of each forbidden assignment to X.

Equivalence classes

Given a set of constraints with set X of common variables, define equivalence classes of assignments to X.

Two assignments are *equivalent* if they extend in the same way.

We can replace X with a single variable whose domain is the set of equivalence classes.

If these can be computed in polynomial time, this reduction is in polynomial time too.

Dual of a hypergraph

Hypergraph of CSP: Variables as vertices, constraint scopes as hyperedges.

We want to identify variables that occur in the same hyperedges.

The *dual* of a hypergraph has a vertex for every hyperedge, and a hyperedge for every vertex.

Taking the dual *twice* replaces variables that occur in the same hyperedges with a single variable.

The dual, in pictures

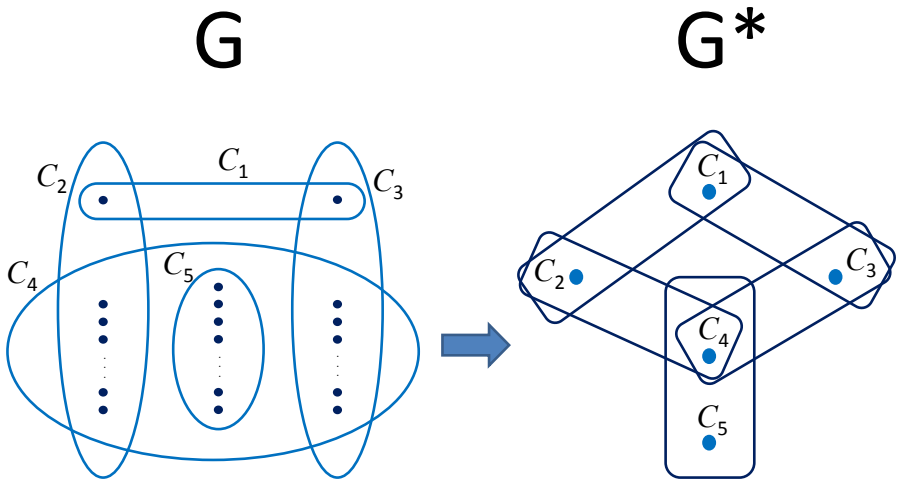

Figure: Hypergraph G and dual G*.

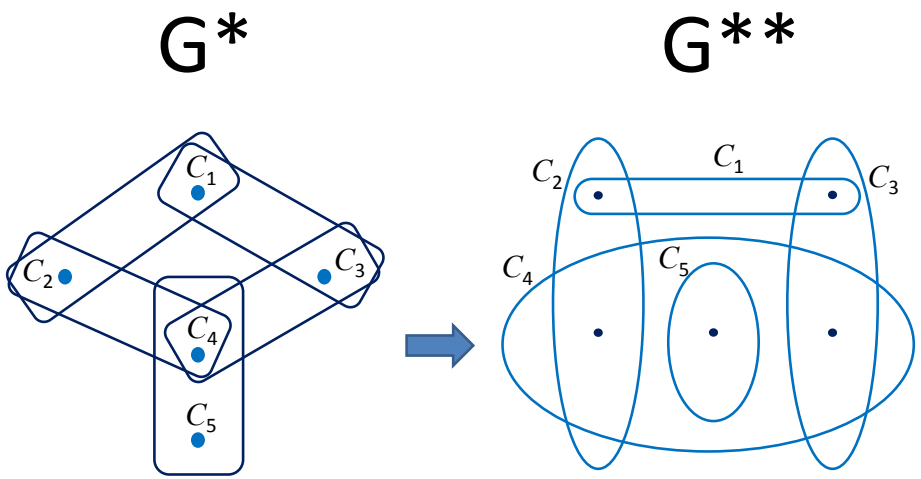

Figure: G* and its dual.

New class

Treewidth of the dual of the dual (twDD).

Only tractable for *cooperating* classes of constraints, that is, those where we can find equivalence classes in polynomial time.

Lies between treewidth and query width.

More about twDD

For SAT, one tractable class is bounded treewidth of the *incidence graph*.

twDD is incomparable with incidence width, so this is a new class for SAT.

Summary

New tractable class for CSP with global constraints.

Includes several nice global constraints, such as cardinality with fixed domain, SAT clauses, etc.

Can be generalized to a tractable class for MaxCSP (to be done in journal paper).