Beyond LexLeader: breaking symmetry with other orderings

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Symmetry breaking

- Static (and many dynamic) methods select one solution in each symmetry class
  - Which one?

Symmetry breaking

- Static (and many dynamic) methods select one solution in each symmetry class
  - Smallest

Symmetry breaking

- Static (and many dynamic) methods select one solution in each symmetry class
  - Smallest in what ordering?

Symmetry breaking

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  - But we could use any other ordering!
  - E.g. GrayCodeLeader = smallest in Gray Code ordering
    - Propose O(n) DC propagator for this!

Gray code ordering

0000
0001
0011
0010
0110
0111
0101
0100
1100
1101
1111
1110
1010
1011
1001
1000

Used in error correcting codes

Gray code ordering

0000
0001
0011
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Breaking symmetry with other orderings

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Breaking symmetry with other orderings

- Complexity
  - Finding LexLeader is NP-hard
  - Can we reduce complexity by choosing a “better” ordering?
- Two step argument
  - A different ordering can increase the complexity of symmetry breaking from polynomial to NP-hard
  - Under modest assumptions (satisfied by Lex Leader, Gray Code, Snake Lex ..), cannot reduce complexity from NP-hard

Worse complexity than LexLeader

THM: Exists polynomial to check ordering $\leq_s$ and problem class P with symmetry group Σ such that for S in P:

$\{S \text{ u } [X_1,..,X_n] \leq_{lex} [X_{\sigma(1)},..,X_{\sigma(n)}] \mid \sigma \text{ in } \Sigma\}$ is polynomial to solve

$\{S \text{ u } [X_1,..,X_n] \leq_s [X_{\sigma(1)},..,X_{\sigma(n)}] \mid \sigma \text{ in } \Sigma\}$ is NP-hard to solve

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$\{S \cup [X_1,..,X_n] \leq_s [X_{\sigma(1)},..,X_{\sigma(n)}] \mid \sigma \in \Sigma\}$ is NP-hard to solve

NB symmetry group exponentially large

Any poly sized group is polynomial to break

No better complexity than LexLeader

THM: Given any simple ordering ≤, there exists symmetry group Σ such that finding smallest assignment in its symmetry class is NP-hard

No better complexity than LexLeader

THM: Given any **simple ordering** ≤, there exists symmetry group Σ such that finding smallest assignment in its symmetry class is NP-hard

Simple ordering:

Compute position in ordering of any assignment in poly time

Given any position, can compute assignment at that position in poly time

No better complexity than LexLeader

THM: Given any **simple ordering** ≤, there exists symmetry group Σ such that finding smallest assignment in its symmetry class is NP-hard

Simple ordering:

Lex ordering

Gray code ordering

Snake Lex ordering

....

No better complexity than LexLeader

THM: Given any **simple ordering** ≤, there exists symmetry group Σ such that finding smallest assignment in its symmetry class is NP-hard

Hence, breaking symmetry with Gray Code, Snake Lex … is also intractable

Breaking matrix symmetry

Frequently occurring special case

Already intractable

THM: Finding smallest solution up to row & col symmetry within LexLeader, Snake-Lex or Gray code ordering is NP-hard

Experimental results

- Promising experimental results using other orderings besides LexLeader
  - Need to align symmetry breaking with branching heuristic
  - Need to align symmetry breaking with optimisation objective
    - Eg. Maximize objective => look for largest in each symmetry class

Experimental results

- 3 problem domains (all optimisation problems)
  - Maximum density still life
  - Low autocorrelation binary sequences
  - Peaceable armies of queens
- Symmetry breaking
  - None, (anti-)lex, (anti-)gray
  - Left2Right, Right2Left, InsideOut, OutsideIn, Row-wise, Col-wise, Snake, SpiralIn, SpiralOut, ...
- Branching heuristic
  - Left2Right, Right2Left, Row, Col, FF, Degree, Constr, Snake, SpiralIn, SpiralOut ...

Peaceable armies of queens

Sym breaker	N=6	N=7	N=8
none	37,434	679,771	19,597,858
gray col	9,763	214,291	5,008,279
gray row	6,975	116,725	3,705,591
lex row	6,880	115,999	3,652,269
anti-gray row	4,326	105,837	2,357,024
anti-lex col	4,373	69,484	2,291,512
anti-gray col	4,317	70,632	1,698,492

Conclusions

- Symmetry breaking is intractable in general
  - Not just with lexicographical ordering
  - But with any (reasonable) total ordering on assignments
- Breaking symmetry with other orderings (e.g. Gray code)
  - Offers some promise in practice
  - Align symmetry breaking better with branching heuristic & objective function

Questions?