A Characterisation of the Complexity of Forbidding Subproblems in Binary Max-CSP

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2012, October 10th

Motivation behind forbidden subproblems

General problem: **Tractable Classes in Max-CSP**

Classical approaches

- Restrictions on the constraints.
- Restrictions on the graph.

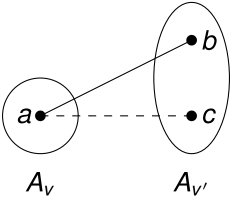
Recent method: **Forbidden Subproblems**
Allows the combination of constraint-based and graph-based approaches.

Subproblem $\equiv$ Instance

- A set of variables.
- A set of points (variable/value assignments).
- A cost on the edges (pairs of points).

Example

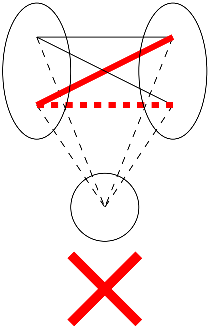
- A set of variables: $\{v, v'\}$.
- A set of points: $\{a \in A_v\} \cup \{b, c \in A_{v'}\}$.
- A cost on the edges: Cost(ab)=0, Cost(ac)=1.



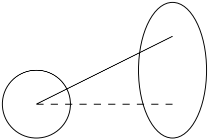
A subproblem P

Forbidding a subproblem

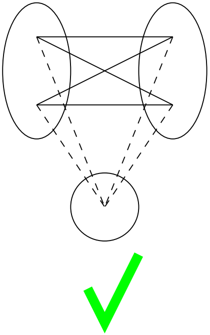
$\mathcal{F}(P)$: Set of Max-CSP instances in which the subproblem P does not occur.



P occurs
in this instance



P



P does not occur
in this instance

P tractable $\Leftrightarrow$ $\mathcal{F}(P)$ tractable

Suppose that P occurs in P':

$$P \text{ does not occur in } I \Rightarrow P' \text{ does not occur in } I$$
$$\mathcal{F}(P) \subset \mathcal{F}(P')$$
$$\mathcal{F}(P) \text{ is tractable} \Leftarrow \mathcal{F}(P') \text{ is tractable}$$
$$\mathcal{F}(P) \text{ is intractable} \Rightarrow \mathcal{F}(P') \text{ is intractable}$$

Lemma

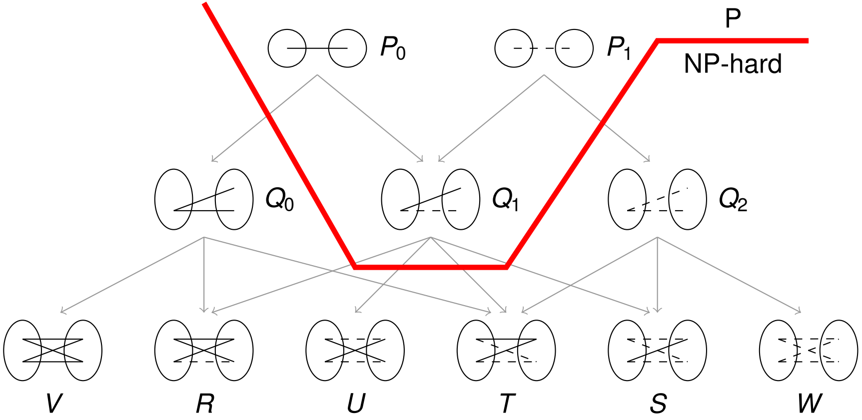
Let P be a subproblem with three or more values in the domain of some variable. Then $\mathcal{F}(P)$ is intractable.

Proof

Reduction from Max-Cut:

1. Max-Cut is intractable.
2. Any Max-Cut instance can be reduced to a Max-CSP instance on boolean domains.
3. $\mathcal{F}(P)$ includes all Max-CSP instances on boolean domains.
4. $\mathcal{F}(P)$ includes all Max-Cut instances (after reduction).
5. $\mathcal{F}(P)$ is intractable.

Two-variable subproblems



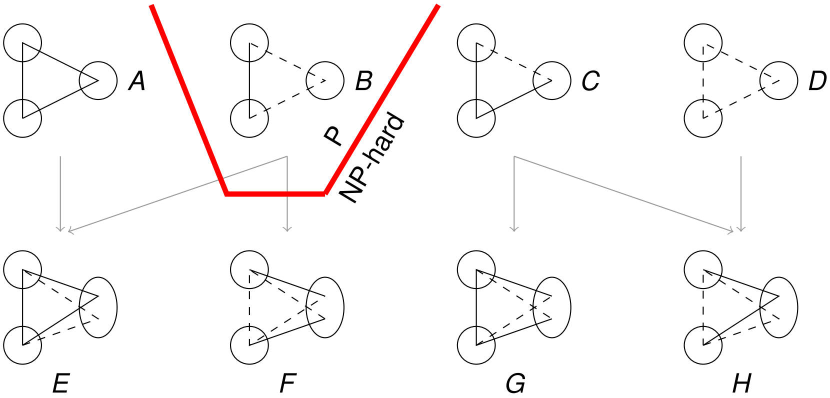
Lemma

1. $\mathcal{F}(Q_1)$ is tractable.
2. $\mathcal{F}(Q_0)$, $\mathcal{F}(Q_2)$ and $\mathcal{F}(U)$ are intractable.

Proof

1. Reduction to a problem where all constraints are constant cost functions.
2. Reduction from Max-Cut.

Three-variable subproblems not containing any of Q_0, Q_2 or U:



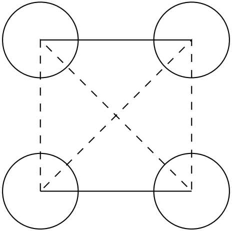
Lemma

1. $\mathcal{F}(B)$ is tractable.
2. $\mathcal{F}(A)$, $\mathcal{F}(C)$ and $\mathcal{F}(D)$ are intractable.
3. $\mathcal{F}(F)$ is intractable.

Proof

1. Cooper and Živný 2011.
2. Cooper and Živný 2011.
3. Reduction from Max-Cut.

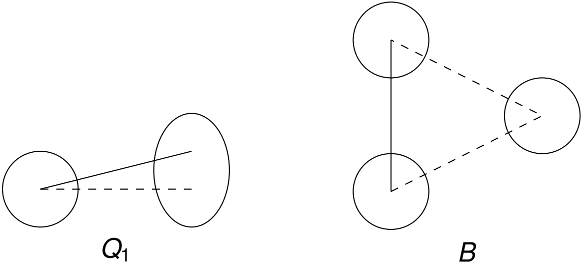
The only subproblem on four or more variables, and not containing any NP-hard subproblem on three or less variables



is NP-hard by a reduction from Max-Cut.

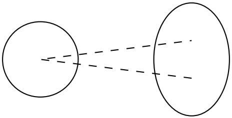
Theorem

If P is a binary Max-CSP subproblem, then $\mathcal{F}(P)$ is tractable if and only if P occurs either in Q_1 or in B.



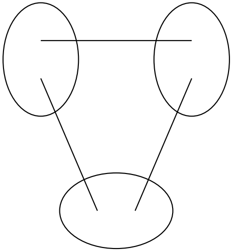
Definitions

- A boolean subproblem: a subproblem with domains of size at most 2.
- A negative (positive) edge pair: Two edges of cost 1 (of cost 0) in a same constraint.



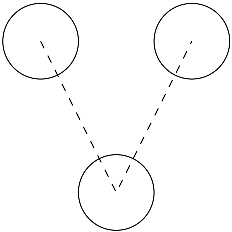
Negative edge pair

- A negative (positive) cycle: A set of k variables $v_1, \ldots, v_k$ such that there is an edge of cost 1 (of cost 0) in the constraint between v_i and v_{i+1} for $1 \leq i \leq k-1$ as well as in the constraint between v_k and v_1.



Positive cycle

- A negative (positive) pivot: Three points p, q, r such that Cost(pq)=Cost(pr)=1 (=0).



Negative pivot

Proposition

If Σ is a finite set of subproblems, then $\mathcal{F}(\Sigma)$ is tractable only if:

1. There is a boolean subproblem $P \in \Sigma$ such that P contains no negative edge pair, no negative cycle and at most one negative pivot.
2. There is a boolean subproblem $Q \in \Sigma$ such that Q contains no positive edge pair, no positive cycle and at most one positive pivot.
3. There is a boolean subproblem $R \in \Sigma$ such that R contains neither Q_0 nor Q_2.

Q_0 Q_2

Summary of Results

1. Dichotomy when forbidding a single subproblem.
2. Necessary conditions when forbidding sets of subproblems.

Thank you very much for your attention!

Please ask questions if you have any.