Contributions to the Theory of Practical Quantified Boolean Formula Solving

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http://www.cse.ucsc.edu/∼avg/

http://www.cse.ucsc.edu/∼avg/QPUP/
These slides are `qpup-cp12-trans.pdf`

http://www.cse.ucsc.edu/∼avg/ProofChecker/
Software directory, contains `QdpllexpSimple.tar`.

Topic: Quantified Boolean Formulas (QBFs)

Published search procedure (QDPLL) finds a conflict in linear time [Giunchiglia *et al.*, JAIR 2006].

First result (Section 3):

To learn ***one clause*** from this conflict may require exponential time.

Construction Idea:

A directed acyclic graph (DAG) can have exponentially many paths.

Topic: Quantified Boolean Formulas (QBFs)

Second result (Section 4):

QBF Pseudo-Unit Propagation (QPUP) guarantees to learn ***one clause***
from a conflict in *polynomial time*.

Construction Idea:

Use **forward reasoning**.

Topic: Quantified Boolean Formulas (QBFs)

Q-resolution is refutationally complete
[Kleine Büning *et al.*, Information and Computation 1995].

Q-resolution is the reasoning system underlying most QBF solvers.

Q-resolution is ***not*** inferentially complete.

This paper introduces **QU-resolution** as an extension of Q-resolution.

Third result (Section 5):

QU-resolution ***is*** inferentially complete.

Fourth result (Section 5):

A refutation in QU-resolution can be exponentially shorter than the shortest
Q-resolution refutation.

Topic: Quantified Boolean Formulas (QBFs)

This paper introduces **depth monotonic** and **depth nondecisive** literals.

An existential literal q is **depth monotonic** if it is of innermost scope in all clauses in which it occurs, but $\overline{q}$ might not have this property.

A literal q is **depth nondecisive** if it is depth monotonic except for some clauses that are not important in a certain technical sense.

Fifth result (Section 6):

Variable-Elimination Q-resolution (QVER) may be performed on depth nondecisive literals without changing the truth value of the QBF.

What are Quantified Boolean Formulas (QBFs)?

Most general definition:

- Add quantification ($\forall u$, $\exists e$) as a new propositional operation.
- A quantified variable must be *true* or *false*.

Least general definition:

- $\mathcal{F}$ is a quantifier-free propositional formula in *conjunctive normal form*.
- $\overrightarrow{Q}$ is a sequence of quantified variables, outer to inner scopes.
- $\Phi = \overrightarrow{Q}\,.\,\mathcal{F}$ is a ***prenex CNF formula*** (QCNF).
- This paper considers *closed* QCNF (all variables quantified).
- Example (chart form):

Φ	$\forall u$	$\forall v$	$\exists e$	$\exists f$	$\forall w$	$\exists d$
C_1	u			f	w	$\overline{d}$
C_2	$\overline{u}$	$\overline{v}$	$\overline{e}$		$\overline{w}$	$\overline{d}$
C_3		v	e	f	w	d
$\ldots$						

Literal Naming Convention

- Lowercase letters near the beginning of the alphabet are ***existential*** literals (or variables, if specified in the context), e.g., c, d, e, etc.
- Lowercase letters near the end of the alphabet are ***universal*** literals (or variables, if specified in the context), e.g., t, u, v, etc.
- p, q, r, s are of unspecified quantifier type.
- $|p|$ denotes the *variable* underlying the *literal* p. (Mainly for quantifier sequences.)

QBF as a Two-Player Game

Two players: A is *Universal*, E is *Existential*.

$\Phi = \overrightarrow{Q}\,.\,\mathcal{F}$ is a PCNF (prefix: $\overrightarrow{Q}$, matrix: $\mathcal{F}$).

When outermost (unassigned) variable is ***universal***, A chooses a value for it and Φ gets simplified.

When outermost (unassigned) variable is ***existential***, E chooses a value for it and Φ gets simplified.

If Φ simplifies to *false*, A wins. If Φ simplifies to *true*, E wins.

"Definition": ***Truth-Value Semantics*** of Φ (coarse grain):

- The value of Φ is *false* (or 0) if and only if A has a winning strategy.
- The value of Φ is *true* (or 1) if and only if E has a winning strategy.

QPUP exponentially shorter than QDPLL Learning

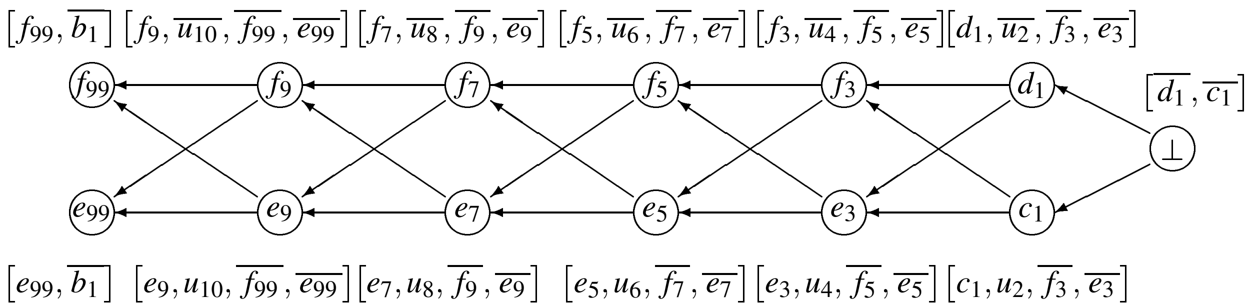

Figure shows conflict graph involving half of index 6 of the hard family.

- b_1 was assumed.
- Other half of the formula has clauses with b_1 and different variables.

After assuming b_1, QDPLL learning [Giunchiglia *et al.*, JAIR 2006] requires 126 resolutions to derive $[\overline{b_1}]$.

- Works right to left. Universal reductions are difficult to set up.

QPUP requires 10. Works left to right. Universal reductions are easy to set up.

Ratio approximately doubles for each succeeding index.

Experimental data in the proceedings.

Conclusion

QPUP avoids exponentially long derivations of a single learned clause.

QU-resolution provides a means to derive more clauses than Q-resolution.

QU-resolution sometimes provides exponentially shorter derivations than Q-resolution.

Recognizing depth-monotonic literals allows solvers to do more simplifications.

Better understanding of QBF theory.